Stable two-dimensional solitary pulses in linearly coupled dissipative Kadomtsev-Petviashvili equations

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We present a two-dimensional (2D) generalization of the stabilized Kuramoto-Sivashinsky system, based on the Kadomtsev-Petviashvili (KP) equation including dissipation of the generic [Newell-Whitehead-Segel (NWS)] type and gain. The system directly applies to the description of gravity-capillary waves on the surface of a liquid layer flowing down an inclined plane, with a surfactant diffusing along the layer's surface. Actually, the model is quite general, offering a simple way to stabilize nonlinear media, combining the weakly 2D dispersion of the KP type with gain and NWS dissipation. Other applications are internal waves in multilayer fluids flowing down an inclined plane, double-front flames in gaseous mixtures, etc. Parallel to this weakly 2D model, we also introduce and study a semiphenomenological one, whose dissipative terms are isotropic, rather than of the NWS type, in order to check if qualitative results are sensitive to the exact form of the lossy terms. The models include an additional linear equation of the advection-diffusion type, linearly coupled to the main KP-NWS equation. The extra equation provides for stability of the zero background in the system, thus opening a way for the existence of stable localized pulses. We focus on the most interesting case, when the dispersive part of the system is of the KP-I type, which corresponds, e.g., to capillary waves, and makes the existence of completely localized 2D pulses possible. Treating the losses and gain as small perturbations and making use of the balance equation for the field momentum, we find that the equilibrium between the gain and losses may select two steady-state solitons from their continuous family existing in the absence of the dissipative terms (the latter family is found in an exact analytical form, and is numerically demonstrated to be stable). The selected soliton with the larger amplitude is expected to be stable. Direct simulations completely corroborate the analytical predictions, for both the physical and phenomenological models.

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I. INTRODUCTION AND DERIVATION OF THE MODEL

Localized structures, such as solitary pulses (SPs), play a dominant role in many conservative and dissipative nonlinear physical systems. As is commonly known, in conservative systems SPs are supported by a balance between nonlinearity and dispersion [1], while in dissipative models, such as the Ginzburg-Landau equations, it must be supplemented by the balance between losses and gain [2].

An important example of a one-dimensional (1D) model that combines conservative and dissipative effects is a mixed Kuramoto-Sivashinsky (KS)—Korteweg–de Vries (KdV) equation, which was first introduced by Benney [3] and is therefore also called the Benney equation. This equation finds various applications in plasma physics, hydrodynamics, and other fields [4,5]. SPs are, obviously, important objects in systems of this type [6]; however, they cannot be completely stable objects in the Benney equation, as the zero solution, which is a background on top of which SPs are to be found, is linearly unstable in this equation due to the presence of the linear gain (however, if the dispersion part of the Benney equation is large enough, the growing perturbation, moving at its group velocity, does not actually overlap with the SP and therefore does not destroy it; see Refs. [7] and references therein). A stabilized version of the Benney equation was recently proposed in Ref. [8]. It is based on the KS-KdV equation for a real wave field $u(x,t)$, which is linearly coupled to an additional linear equation of the diffusion type for an extra real field $v(x,t)$, which provides for the stabilization of the zero background:

$$u_t+uu_x+u_{xxx}-v_x=-\alpha u_{xx}-\gamma u_{xxxx}, \quad (1)$$

$$v_t+cv_x-u_x=\Gamma v_{xx}. \quad (2)$$

Here, α, γ, and Γ are coefficients accounting for the gain and loss in the u subsystem and loss in the v subsystem, respectively, and c is a group-velocity mismatch between the fields.

It was shown both analytically (by means of the perturbation theory) and numerically in Ref. [8] that the system (1), (2) gives rise to a completely stable SP, as well as to stable bound states of SPs, in a broad parametric region. As a matter of fact, Eqs. (1) and (2) furnish an example of a model of the KS type that gives rise to fully stable pulses, and they can be easily observed in experiment (see a detailed discussion of physical realizations of the model—first of all, in terms of liquid films flowing down under the action of gravity—in the following section).

The liquid films and other systems in which the observation of stable SPs is expected (e.g., double-front flames, see below) are 2D media, therefore a relevant issue is to introduce a physically meaningful 2D version of Eqs. (1) and (2) and seek for stable 2D pulses in the generalized model. This is the objective of the present work. It will be demonstrated that the 2D model which will be derived here is a generic one for a number of different applications. The results will directly point at a type of 2D pulses that can be observed experimentally in a straightforward way. Besides that, a possibility of the existence of stable 2D pulses may help to understand the phenomenon of turbulent spots; see, e.g., Refs. [9] and references therein.

The paper is organized as follows. The 2D model is derived in detail in Sec. II, starting with a particular physical problem, viz., a downflowing liquid film carrying a surfactant, and then proceeding to the generic form of the model. Parallel to the derived model, we will also consider its counterpart, which differs by the form of 2D dissipative terms, in order to demonstrate that basic results are insensitive to the particular features of the model (which is relevant to show, even if the model is generic). In Sec. III, we consider the stability of the zero solution in the 2D model, which, as well as in the 1D case, is a necessary condition for the full stability of SPs. In Sec. IV, an analytical perturbation theory for SPs is developed by treating the gain and loss constants as small parameters. To this end, a family of exact 2D soliton solutions of the zero-order system (the one without the gain and loss terms) is obtained, following the pattern of the well-known ‘‘lump’’ solitons of the Kadomtsev-Petviashvili-I (KP-I) equation. Then, using the balance equation for the net field momentum, similar to how it was done in the 1D model [8], we demonstrate that the combination of the gain and loss terms may select two (or no) stationary pulses out of the continuous soliton family existing in the zero-order system; in the case when two stationary pulses are found, it is very plausible that the one with the larger value of its amplitude is stable. In Sec. V, we present results of direct numerical simulations of the full 2D model, which completely confirm the analytical predictions, i.e., the existence of stable 2D localized SPs. The paper is concluded by Sec. VI.

II. THE MODEL

The physical meaning of the model based on Eqs. (1) and (2), and its 2D generalization developed below, can be understood in terms of a particular application to a thin downflowing liquid layer with a surfactant trapped on its surface. As is well established (see a review [5]), in the 1D case the evolution of the flow velocity field $u(x,t)$ in the layer is governed, in appropriately chosen units, by the KS-KdV (Benney) equation $u_t+uu_x+u_{xxx}=-\alpha u_{xx}-\gamma u_{xxxx}$. In this equation, the gain α is induced by gravity, and the loss parameter γ is proportional to the fluid's viscosity coefficient, while the left-hand side is generated by Euler's equations (for an irrotational flow), exactly the same way as in the classical derivation of the KdV equation. If the surfactant is distributed on the surface of the layer with a density $c+v(x,t)$, where c and v are, respectively, its constant and small variable parts ($|v|\ll c$), the gradient of v creates, through the variation of the surface tension, an additional force $\sim v_x$ which drives the flow, hence the Euler equation adds the term v_x to the right-hand side of the KS-KdV equation (the coefficient in front of this term may be scaled to be 1), so that the equation takes precisely the form (1). Further, the evolution of the surfactant density is governed by an obvious advection-diffusion equation: $v_t+[u(c+v)]_x=\Gamma v_{xx}$, where Γ is the surface diffusion constant. With regard to the condition $|v|\ll c$, the latter equation takes the form (2).

In the 2D case, we consider a quasi-1D (weakly 2D) flow of the film, with the y scale much larger than that along the x axis. In other words, if the wave is taken as $\exp(iKx+iQy)$, the small wave numbers are ordered so that

$$Q\sim K^2. \tag{3}$$

Then, according to the classical derivation [10], the KdV part of Eq. (1) is replaced either by the KP-I equation,

$$(u_t+uu_x+u_{xxx})_x=u_{yy}, \tag{4}$$

or by the KP-II equation, which is

$$(u_t+uu_x+u_{xxx})_x=-u_{yy}, \tag{5}$$

the coordinate y being properly rescaled. The choice between Eqs. (4) and (5) is determined by the sign of the 2D correction to the dispersion; in particular, the capillarity gives rise to the KP-I equation. The difference between the KP-I and KP-II equations is that, although both of them have quasi-1D (i.e., y-independent) soliton solutions that reduce to the usual KdV solitons, only in the KP-II equation this soliton is stable against y-dependent perturbations. On the other hand, the KP-I equation has stable 2D solitons (the so-called ‘‘lumps’’), which are weakly (nonexponentially) localized in both x and y, see below; the KP-II equation does not have 2D solitons.

The next step is to accordingly generalize, for the 2D situation, the dissipative term in Eq. (1). Dissipative generalizations of the KP equations were introduced in some works, see, e.g., Refs. [11]. Those generalizations follow the pattern of the arrangement of the KP equations proper: if one starts from a corresponding 1D equation containing dissipative terms [for instance, Eq. (1)], which is written as something$=0$, its 2D counterpart is $(\text{something})_x=\pm u_{yy}$. The accordingly modified equations (1) and (2) then take the form

$$(u_t+uu_x+u_{xxx}-v_x+\alpha u_{xx}+\gamma u_{xxxx})_x=\pm u_{yy}, \tag{6}$$

$$(v_t+cv_x-u_x-\Gamma v_{xx})_x=\pm v_{yy}. \tag{7}$$

The second equation in this system can be simplified, as the usual ordering of the partial derivatives, adopted in course of the quasi-1D derivation [10], implies that v_{yy} is a small quantity of a higher order than v_{xxx}; hence v_{yy} may be dropped, so that Eq. (7) remains one dimensional. As a result, this version of the 2D system takes the form

$$(u_t+uu_x+u_{xxx}-v_x+\alpha u_{xx}+\gamma u_{xxxx})_x=\pm u_{yy}, \tag{8}$$

$$v_t+cv_x-u_x-\Gamma v_{xx}=0. \tag{9}$$

However, while keeping Eq. (9) in the 1D form is quite acceptable, the way the dissipative and gain terms in Eq. (8) were made two dimensional was formal, not being based on any physical argument. Moreover, it will be shown in the following section that, unlike its 1D counterparts (1) and (2), this 2D model cannot produce any stable solitary pulses, as its zero solution is always unstable.

In order to derive a physically relevant form of the dissipative part of the u equation, one should resort to the standard procedure that derives a *generic* set of dissipative terms in the quasi-1D situation (in the context of convective flows) in the Newell-Whitehead-Segel (NWS) equation [12]. This equation gives a simple prescription, which, as in the KP equations, is based on the ordering (3) of the longitudinal and transverse wave numbers: the longitudinal dissipative term γu_{xxxx} must be supplemented by its transverse counterpart γu_{yy} [the scaling of the transverse coordinate y in the NWS equation is precisely the same, which casts the KP equation in the standard form (4) or (5); the identity of the scalings is not accidental, being a consequence of the fact that both KP and NWS equations are generic ones in the quasi-1D geometry, the former one in the class of dispersive equations, and the latter among dissipative equations]. Thus, the proper form of the 2D system is

$$(u_t+uu_x+u_{xxx}-v_x+\alpha u_{xx}+\gamma u_{xxxx}-\gamma u_{yy})_x=\pm u_{yy}, \tag{10}$$

$$v_t+cv_x-u_x-\Gamma v_{xx}=0. \tag{11}$$

In fact, exactly the same combination of dissipative terms as in Eq. (10) has been derived earlier in asymptotic equations governing nonlinear waves on thin downflowing liquid films [13,14] and in two-fluid flows [15].

Note that Eqs. (10) and (11) do not contain any additional free parameter in comparison with the 1D system (1), (2). As a matter of fact, this is another consequence of the fact that both the dispersive and dissipative parts of the system were derived for the generic quasi-1D case.

A special form of a 2D quasi-isotropic (rather than quasi-1D) generalization of the Benney equation was also derived, which, for instance, describes Rossby waves in a rotating atmosphere [16] (see also Refs. [13,14]):

$$u_t+uu_x+\Delta u_x+\alpha u_{xx}+\gamma\Delta^2 u=0, \tag{12}$$

where $\Delta\equiv\partial_x^2+\partial_y^2$, hence Δ^2 in Eq. (12) is a fourth-order isotropic dissipative operator. Following the pattern of Eq. (12), a quasi-isotropic generalization of Eqs. (1) and (2) may be introduced, replacing the term u_{xxxx} in Eq. (8) by Δ^2, which leads to a system

$$(u_t+uu_x+u_{xxx}-v_x+\alpha u_{xx}+\gamma\Delta^2 u)_x=\pm u_{yy}, \tag{13}$$

$$v_t+cv_x-u_x-\Gamma v_{xx}=0. \tag{14}$$

Note that all the 2D systems introduced above conserve two "masses,"

$$M=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}u(x,y)dx\,dy,\quad N=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}v(x,y)dx\,dy \tag{15}$$

(which indeed have the meaning of masses in the application to liquid-film flows, in the cases of both the single layer with surfactant and two layers with the lower one dominated by viscosity).

Thus, Eqs. (10) and (11) can be derived in a consistent way as a system describing the downflow of a liquid viscous film carrying a surfactant. Actually, the derivation outlined above clearly suggests that this model is a generic one for weakly 2D systems combining dispersion, gain, and viscosity. In particular, a derivation involving more technicalities and following the lines of Ref. [5] shows that the same model applies to a downflow of a two-layered liquid film in the case when the substrate layer is dominated by viscosity. A physically different example may be a double-front flame propagating in a combustible gaseous mixture, in the well-studied case when the combustion involves two consecutive reactions (see Refs. [17] and references therein). In the case when a single-flame front is unstable, it is well known that its evolution is governed by the KS equation proper [18]. It is also known that a situation with one front stable and one unstable is possible, which is described by a linearly coupled system, consisting of a KS equation and the one tantamount to Eq. (13). Dispersion, which is missing in the KS equation proper, can be induced by a background shear flow tangent to the flame [19], but a detailed derivation of the full model for this case is beyond the scope of this work.

As concerns the model (13), (14), we consider it as a semiphenomenological one, that may apply to cases which are "more isotropic" than those obeying the condition (3). We will study this model parallel to the physical one, Eqs. (10), (11), in order to see if qualitative results are sensitive to the details of a given model. Accordingly, the systems (10), (11) and (13), (14) will be referred to below as physical and phenomenological ones, respectively. In the analysis presented in the later sections, we focus on the case of the KP-I type, i.e., with the upper sign in Eqs. (8), (10), and (13), as only in this case one may expect the existence of nontrivial 2D pulses, while the models of the KP-II type may only extend the SP found in Ref. [8] into a quasi-1D (y-independent) pulse in two dimensions.

III. THE STABILITY OF THE ZERO SOLUTION

As previously mentioned, completely stable SPs can only exist in a system whose trivial solution, $u=v=0$, is stable, therefore our first objective is to analyze this necessary stability condition. We substitute into the corresponding linearized equations a 2D perturbation in the form $u\sim\exp(ikx+iqy+\lambda t)$, $v\sim\exp(ikx+iqy+\lambda t)$, where k and q are arbitrary real wave numbers of the perturbation, and λ is the corresponding instability growth rate [note that, as all the equations that we are going to consider are written in the scaled form, k and q are not assumed to be specially small, unlike K and Q in Eq. (3)]. This leads to a linearized dispersion equation

$$[k(\lambda - ik^3 - \alpha k^2 + \gamma k^4) - iq^2](\lambda + ick + \Gamma k^2) + k^3 = 0 \tag{16}$$

for the formal model (8), (9), or

$$[k\{\lambda - ik^3 - \alpha k^2 + \gamma(k^2+q^2)^2\} - iq^2](\lambda + ick + \Gamma k^2) + k^3 = 0 \tag{17}$$

for the phenomenological model (13), (14), or

$$[k\{\lambda - ik^3 - \alpha k^2 + \gamma(k^4+q^2)\} - iq^2](\lambda + ick + \Gamma k^2) + k^3 = 0 \tag{18}$$

for the physical model (10), (11). The stability condition states that both solutions of the quadratic equations (16), (17), or (18) must satisfy the inequality

$$\mathrm{Re}[\lambda(k,q)] \leqslant 0 \tag{19}$$

at all the real values of k and q.

As it was already mentioned, the zero solution in the formal model (8), (9) is always unstable, which can be shown as follows: in the case when k is small, while q is $\sim O(1)$, the two roots of Eq. (16) can be expanded as

$$\lambda_1(k) = -ick - \Gamma k^2 + \cdots, \quad \lambda_2 = iq^2/k + \alpha k^2 + \cdots. \tag{20}$$

Obviously, the second root in Eq. (20) yields instability.

The zero solution may be stable in the physical and phenomenological models. Although the direct check of the condition (19) for Eqs. (17) and (18) at all real values of k and q with the four free parameters is a formidable algebraic problem, it is possible to link the stability condition for the physical system to that which was studied in detail for the 1D system (1), (2) in Ref. [8]. An algebraic transformation shows that, if the condition $\mathrm{Re}[\lambda(k)] \leqslant 0$ holds at all real values of k in Eq. (8) of Ref. [8], then the inequality $\mathrm{Re}[\lambda(k,q)] \leqslant 0$ is true at all real values of k and q in Eq. (18), or, in other words, the stability of the zero solution in the 1D case guarantees its stability in the 2D case for the physical model. However, rather than following formal algebra, it is easier to understand this result from Fig. 1, which shows a 3D plot of the instability growth rate $\mathrm{Re}\,\lambda$ vs k and q for a set of typical values of the parameters. Obviously, when k is large, the stability is secured by the higher-order dissipative term in Eq. (10). The most dangerous case is that when k and q are relatively small. It can be seen from Fig. 1 that the growth rate, considered as a function of q, attains its maximum value at $q=0$ if k is very small. As k increases, two additional local maxima of the growth rate appear at nonzero q, and they may be greater than the local maximum at $q=0$. However, since the local maximum at $q=0$ becomes more negative as k increases, the two side maxima remain negative too. Thus, the zero solution to Eqs. (10) and (11) is stable in the same parameter region in which it was found to be stable in the 1D system (1), (2) in Ref. [8].

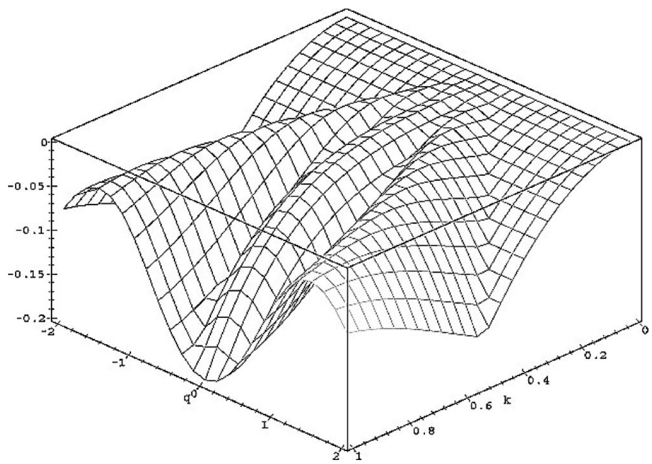

FIG. 1. The instability growth rate $\mathrm{Re}\,\lambda$ for the zero solution in the physical model [Eqs. (10) and (11)] vs the longitudinal and transverse perturbation wave numbers k and q. The values of the parameters are $\alpha=0.2$, $\gamma=0.05$, $c=-1.0$, and $\Gamma=0.55$.

In the phenomenological model, the zero solution is also stable in a certain parametric region. However, no simple relation of the stability condition to that in the 1D system can be found in this model.

IV. THE PERTURBATION THEORY FOR TWO-DIMENSIONAL SOLITARY PULSES

Both the physical and phenomenological models reduce to a zero-order system by setting $\alpha=\gamma=\Gamma=0$, while keeping an arbitrary value of c. This zero-order system is conservative, consisting of the KP-I equation coupled to an extra linear one,

$$(u_t + uu_x + u_{xxx} - v_x)_x = u_{yy}, \quad v_t + cv_x = u_x. \tag{21}$$

Looking for a solution to Eqs. (21) in the form of a soliton traveling at a constant velocity s in the x direction, so that

$$u(x,y,t) = u(\xi,y), v(x,y,t) = v(\xi,y) \quad \text{with} \quad \xi \equiv x - st, \tag{22}$$

we immediately conclude that, as in the 1D case, for such a solution we have

$$v(\xi,y) = (c-s)^{-1} u(\xi,y). \tag{23}$$

With regard to the relation (23), an *exact* solution to Eqs. (21) giving a 2D weakly localized soliton (“lump”) is

$$u(\xi,y) = \frac{24(1+cs-s^2)}{c-s} w(\zeta,z), \tag{24}$$

where

$$z \equiv \frac{1+cs-s^2}{c-s} y, \quad \zeta \equiv \sqrt{\frac{1+cs-s^2}{c-s}}\,\xi,$$

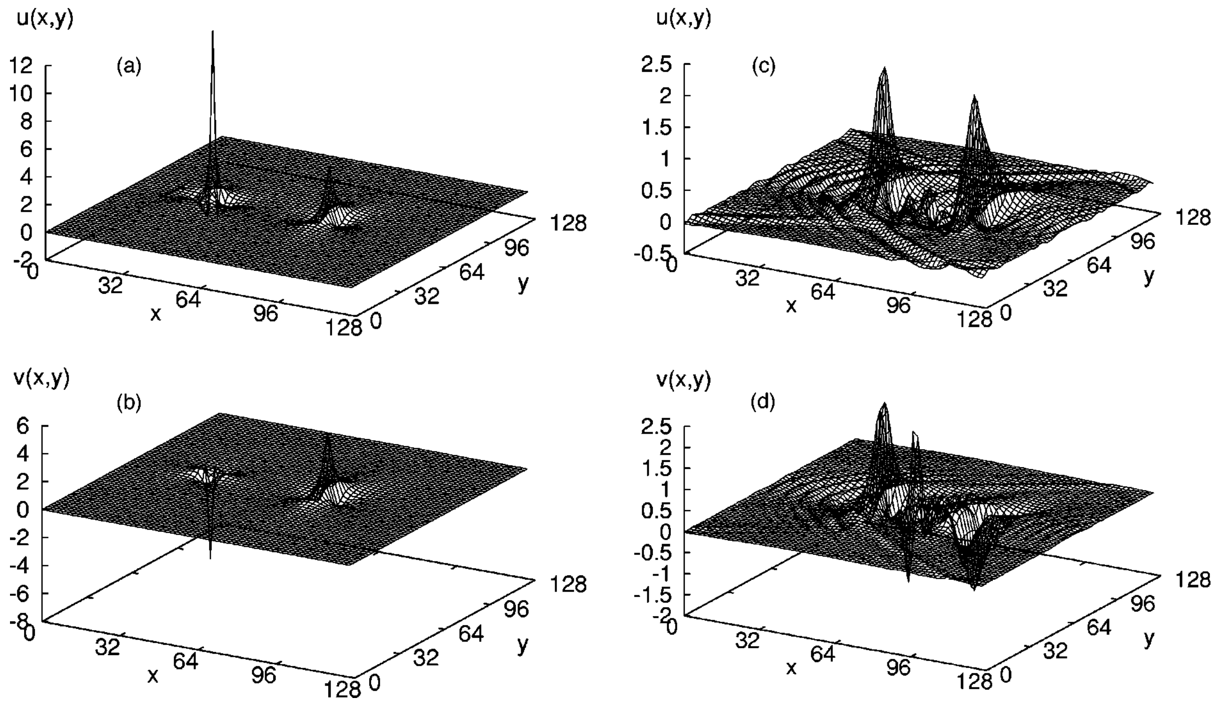

FIG. 2. An example of the inelastic collision between two stable lump-type solitons with different velocities, simulated within the framework of the zero-order conservative system (21) with $c=0$. The u and v fields prior to the collision (at $t=0$) are shown in panels (a) and (b), and after the collision (at $t=40$), in panels (c) and (d). The initial velocities of the two solitons are $s_1=2.0$ and $s_2=-0.8$.

$$w(\zeta,z)\equiv\frac{z^2-\zeta^2+3}{[z^2+\zeta^2+3]^2}. \tag{25}$$

This solution exists in the case

$$\frac{1+cs-s^2}{c-s}>0. \tag{26}$$

Note that, in contrast to the lump soliton solutions of the KP-I equation proper, which may only have positive velocities, the solitons (24) may also move in the negative direction, provided that the velocity satisfies the condition (26). In the particular case $c=0$, this condition means that either $s>1$, or $-1<s<0$.

We have checked by direct simulations of Eqs. (21) that all the lump-type SP solutions are stable within the framework of the unperturbed equations (21). We have also simulated catch-up and head-on collisions between two lump pulses with different velocities. The results clearly show that the collisions are *inelastic*; hence, contrary to the KP-I equation, the conservative system (21) is not an exactly integrable one. An example of the head-on collision between two lump-type SPs with velocities $s=2.0$ and $s=-0.8$ is shown in Fig. 2. The inelastic character of the interaction is obvious.

In fact, the collision shown in Fig. 2 is not quite generic, as it is generated by an initial configuration in which the mass N of the v component, see Eq. (15), is almost zero (N vanishes exactly if the initial velocity of the second soliton is -0.756 instead of -0.8, which is the case in Fig. 2). In this case, a resonance probably occurs, resulting in the generation of an additional pair of solitons, which is suggested by Fig. 2 [especially, by panel 2(d)]. Far from this resonant case, a typical collision (see an example in Fig. 3) is also strongly inelastic, but in this case the result may be regarded, in the first approximation, as the fusion of two solitons into one.

To check the accuracy of the results produced by the direct simulations of the 2D equations, including those displayed in Figs. 2 and 3 and in the following section, the simulations were repeated with the number of points increasing. For instance, Figs. 2 and 3 were obtained in a 2D domain of 256×256 points; repeating the same simulations with 512×512 points, we had obtained the same pictures, without any visible difference.

Next, we restore the terms with the coefficients α, γ, and Γ as small perturbations, with the aim to predict, at the first order of the perturbation theory, which solutions will be selected from the family of the lump solitons (24) found in the zero-order system (21). As it is suggested by the analysis of the 1D system [8], it is convenient to choose the relative velocity, $\delta\equiv c-s$, as a parameter of the family.

As in the 1D case, we select stationary pulses, using the balance condition for the net field momentum,

$$P=\frac{1}{2}\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}dy[u^2(x,y)+v^2(x,y)], \tag{27}$$

which is a dynamical invariant of the conservative (zero-order) system [the masses (15) cannot be used for this purpose, as they remain dynamical invariants in the full dissipative models]. The dissipative and gain terms give rise to the following exact balance equations for the momentum:

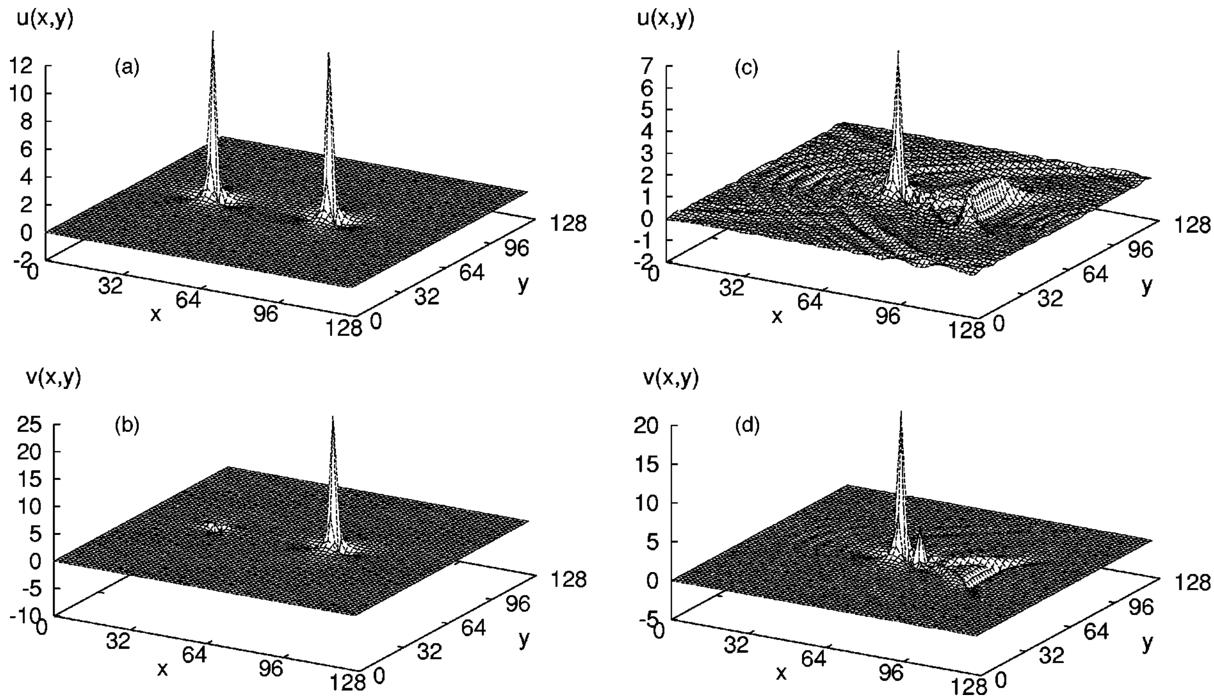

FIG. 3. The inelastic collision in the conservative system (21) with $c=0$ in the case when the initial velocities of the two solitons are $s_1=2.0$ and $s_2=-0.5$. The panels have the same meaning as in Fig. 2, with a difference that the final state is displayed at $t=45$.

$$\frac{dP}{dt}=\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}dy[\alpha u_x^2-\gamma(u_{xx}^2+u_y^2)-\Gamma v_x^2] \quad (28)$$

in the physical system (10), (11), and

$$\frac{dP}{dt}=\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}dy[\alpha u_x^2-\gamma(u_{xx}+u_{yy})^2-\Gamma v_x^2] \quad (29)$$

in the phenomenological system (13), (14).

Substituting the unperturbed exact solution (24) into the equilibrium condition $dP/dt=0$, which follows from Eq. (28) or (29), we cast the equilibrium condition in the form

$$C_1\left(\alpha-\frac{\Gamma}{\delta^2}\right)\left(c-\delta+\frac{1}{\delta}\right)-C_2\left(c-\delta+\frac{1}{\delta}\right)^2=0, \quad (30)$$

where, for both the physical and phenomenological systems,

$$C_1\equiv\int_{-\infty}^{+\infty}dz\int_{-\infty}^{+\infty}d\zeta\left(\frac{\partial w}{\partial\zeta}\right)^2, \quad (31)$$

and the constant C_2 is defined differently for the two models:

$$C_2^{(\mathrm{phys})}=\int_{-\infty}^{+\infty}dz\int_{-\infty}^{+\infty}d\zeta\left[\left(\frac{\partial^2 w}{\partial\zeta^2}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right], \quad (32)$$

$$C_2^{(\mathrm{phen})}=\int_{-\infty}^{+\infty}dz\int_{-\infty}^{+\infty}d\zeta\left(\frac{\partial^2 w}{\partial\zeta^2}+\frac{\partial^2 w}{\partial z^2}\right)^2. \quad (33)$$

Numerical computation of the integrals yields $C_1=0.34708$, $C_2^{(\mathrm{phys})}=0.75984$, and $C_2^{(\mathrm{phen})}=0.85770$. Then, the equilibrium equation (30) yields the following cubic equation for δ (where $\tilde{\alpha}\equiv\alpha/\gamma$, $\tilde{\Gamma}\equiv\Gamma/\gamma$):

$$\delta^3+(0.46\tilde{\alpha}-c)\delta^2-\delta-0.46\tilde{\Gamma}=0, \quad (34)$$

$$\delta^3+(0.41\tilde{\alpha}-c)\delta^2-\delta-0.41\tilde{\Gamma}=0, \quad (35)$$

for the physical and phenomenological models, respectively. Note that these equations have the same general form as the one that selects equilibrium values of the parameter δ in the 1D system investigated in Ref. [8].

Physical roots of Eq. (34) or (35) are defined as those which are not only real, but also satisfy the condition (26). The physical roots select particular lump-type SPs, from the family of lump solitons of the zero-order system, which remain steady pulses in the presence of the perturbations. As in the 1D model, the existence of at least two physical roots is necessary for one pulse to be stable (the other one, with a smaller value of the amplitude, is then automatically unstable) [8]. A numerical solution of Eqs. (34) and (35) verifies that both equations have exactly two physical roots in a broad parametric region, while three physical roots can never be found. Interestingly, the velocities corresponding to these two physical roots are both positive, i.e., a stable lump SP cannot move in the negative x direction.

Regions in the parameter plane (α,Γ) in which exactly two physical roots have been found are shown, for $\gamma=0.05$, in Figs. 4 and 5 for the physical model, with $c=0$ and $c=-1$, respectively, and in Fig. 6 for the phenomenological model with $c=-1$. These regions are bounded by two solid lines in Figs. 4–6. Beneath the lower solid line, Eq. (34) or (35) has three real roots, but no more than one of

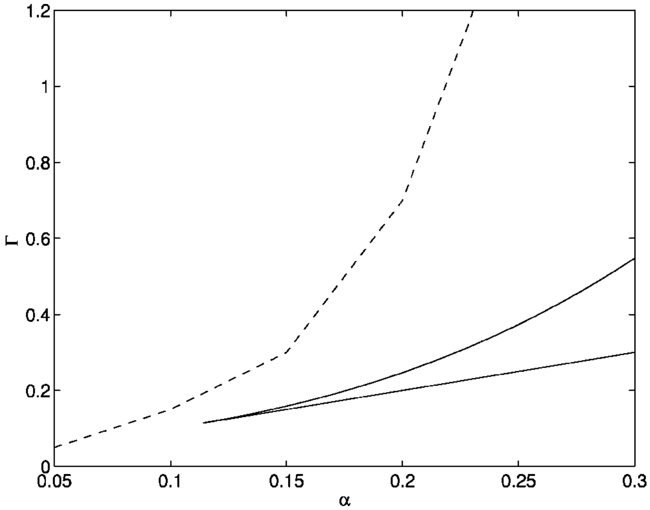

FIG. 4. The stability region of the zero solution, and the region inside which two physical roots exist for Eq. (34) with $\gamma=0.05$ and $c=0$. The zero solution is stable above the dashed line, and inside the region bounded by two solid lines, Eq. (34) produces two physical solutions.

them is physical, and above the upper solid line, two physical roots bifurcate into a pair of complex ones, leaving no physical roots.

The dashed line is the border above which the zero solution is stable in the 1D model (1), (2), hence it is also stable in the 2D physical model, according to the results reported in the preceding section. As seen in Fig. 4, in the parameter plane (α,Γ) of the physical system with $c=0$, there is no overlapping between regions where the zero solution is stable, and where the perturbation theory selects two physical solutions for the 2D pulses, but a narrow overlapping region is found in the same model for $c=-1$, see Fig. 5. It is expected that stable lump-type SPs exist inside this narrow region, which is confirmed by direct simulations, as reported

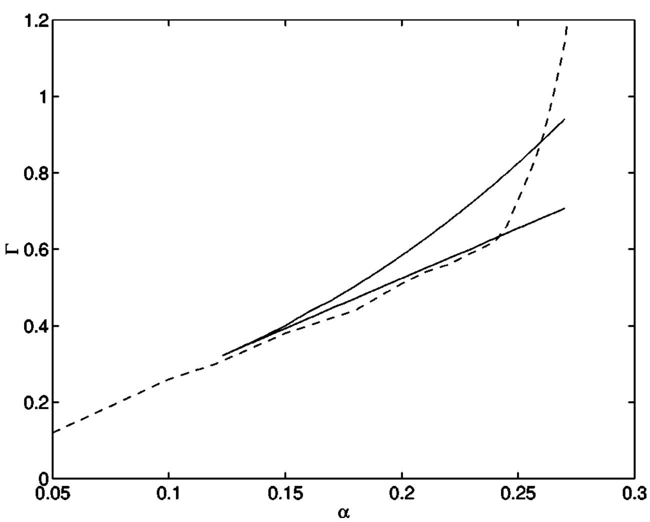

FIG. 5. The expected stability region for the solitary pulses in the parametric plane (α,Γ) for $\gamma=0.05$ and $c=-1.0$. The solid and dashed lines have the same meaning as in Fig. 4.

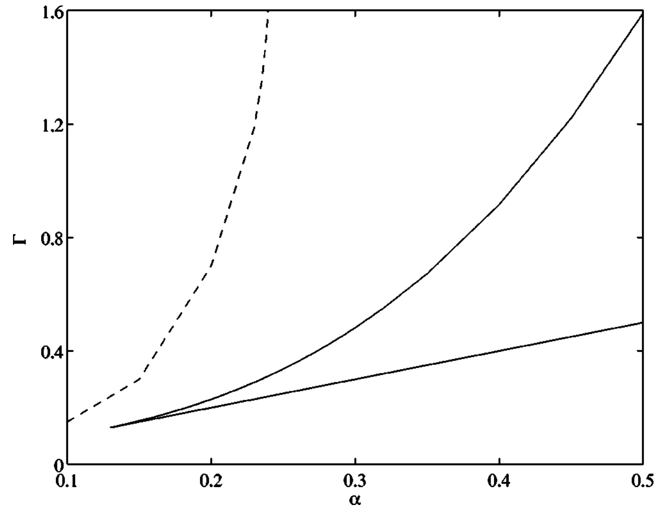

FIG. 6. The stability region of the zero solution of the 1D model (1), (2) and the region inside which two physical roots exist for Eq. (35) when $\gamma=0.05$ and $c=-1$. The solid and dashed lines have the same meaning as in Fig. 4.

in the following section. Despite the absence of an overlapping region in Fig. 6, we have to check the condition (26) for Eq. (17), i.e., the stability of the zero solution in the phenomenological model, point by point, since the stability of the zero solution in the 1D model does not guarantee the stability of the zero solution in the phenomenological system, see the preceding section.

The same analysis can be performed for values of the loss parameter γ other than 0.05, which was fixed in the above consideration. The results show that the variation of γ produces little change in terms of the expected SP stability region. On the other hand, the group-velocity mismatch c affects the stability region significantly. We had numerically found that there is no stability region in both the physical and phenomenological model for $c\geqslant 0$. Then, by inspecting the region of negative c, it was found that there is a critical value c_{cr} such that stable pulses become possible for $c<c_{\mathrm{cr}}$. In the physical model, $c_{\mathrm{cr}}\approx-0.8$.

To conclude the analytical consideration, it is necessary to stress that the stability conditions obtained here are only necessary ones. Obtaining a full set of sufficient stability conditions might be possible within the framework of a rigorous spectral analysis of the full system linearized around the pulse solution, which was done (in a numerical form) for the 1D Benney (KS-KdV) equation in Refs. [7]. Extending this analysis to the two-component 2D case is an extremely difficult problem, which is beyond the scope of the present work. Nevertheless, numerical results reported in the following section strongly suggest that the necessary stability conditions, which were obtained above by means of the simple analytical methods, are, most probably, sufficient for the stability of the 2D pulses.

V. DIRECT SIMULATIONS OF THE TWO-DIMENSIONAL SOLITARY PULSES

To check the predictions made by the above analysis for both the physical and phenomenological models, via direct

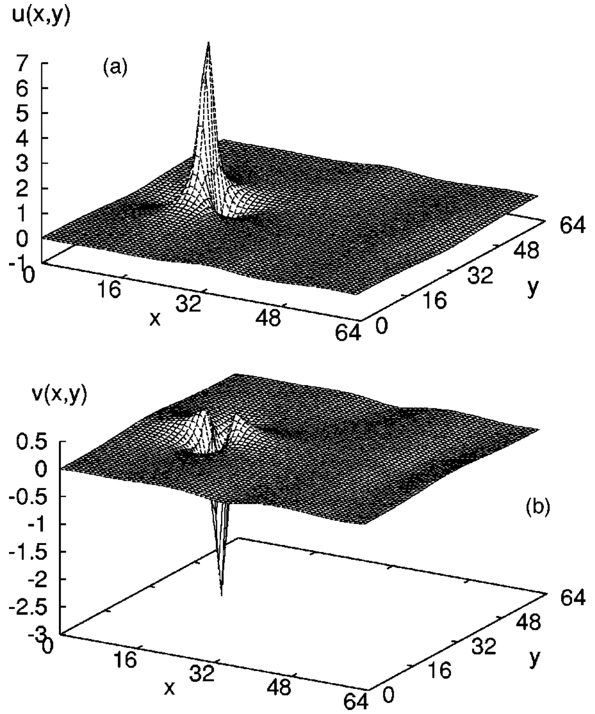

FIG. 7. A stable lump-type solitary pulse in the physical model, found as a result of long numerically simulated evolution in the case $\alpha=0.2$, $\gamma=0.05$, $c=-1.0$, and $\Gamma=0.55$. The panels (a) and (b) show established shapes of the u and v fields, respectively.

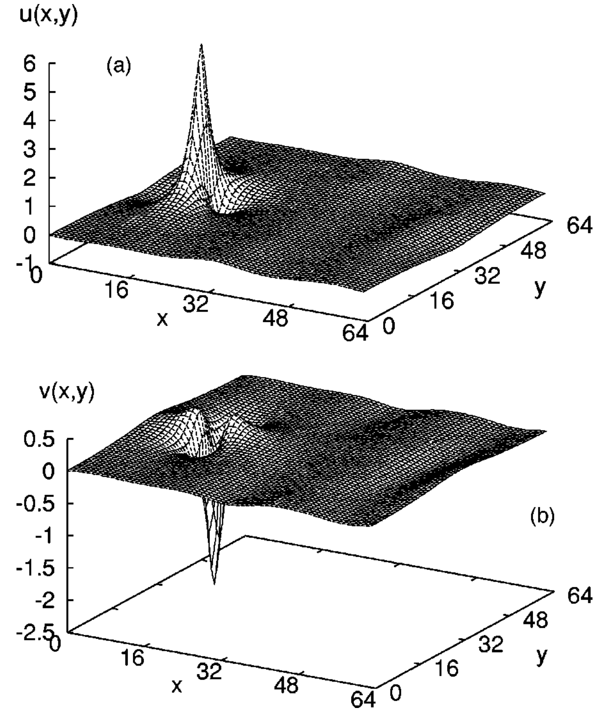

FIG. 8. A stable lump-type solitary pulse in the phenomenological model, found the same way as the pulse shown in Fig. 7 for the same values of the parameters. The panels (a) and (b) have the same meaning as in Fig. 7.

numerical simulations, we have upgraded the implicit pseudospectral scheme, which was developed previously for the 1D model (1), (2) (see Appendix in Ref. [8]), to a scheme capable of dealing with 2D models. It it relevant to mention that results obtained with a rather coarse grid, $\Delta x=\Delta y=0.5$, and relatively large time steps, $\Delta t=0.1$, remained virtually unchanged if reproduced with an essentially finer grid and smaller time step. The initial conditions were taken as the lump-soliton solutions of the zero-order system, but with arbitrarily modified values of the amplitude, in order to check whether strongly perturbed pulses relax to stable ones, i.e., whether the stable pulses are *attractors*.

Collecting data produced by the systematic simulations, it has been found that, for the physical model, stable lump-type SPs do exist and are stable *everywhere* inside the stability region in the (α,Γ) parameter plane, which was predicted by the analytical perturbation theory, see Fig. 5. Moreover, all the stable pulses were found to be strong attractors indeed. A typical lump-type SP with $\alpha=0.2$, $\gamma=0.05$, $c=-1.0$, and $\Gamma=0.55$ is displayed in Fig. 7. The initial-state pulses selected as mentioned above definitely relax to this single stationary lump-type SP, provided that the initial amplitude A_0 of their u component exceeds 1.5. For instance, starting with the initial amplitudes $A_0=1.96$ and $A_0=13.33$ [the corresponding values of the unperturbed solitons' velocity are $s_0=0.8$ and $s_0=2.0$, see Eq. (24)], a lump-type SP develops with the values of the amplitude $A_{\mathrm{num}}=7.11$ and $A_{\mathrm{num}}=7.16$, and velocities $s_{\mathrm{num}}=1.32$ and $s_{\mathrm{num}}=1.33$, respectively, by the time $t=400$. Meanwhile, the analytical prediction for the amplitude and velocity of the presumably stable pulse, given by Eq. (34) for the same values of the parameters, is $A_{\mathrm{anal}}\approx 7.38$ and $s_{\mathrm{anal}}\approx 1.35$. Thus, the numerical results match the theoretical prediction well.

On the other hand, if the initial amplitude is too small, e.g., $A_0=0.89$ $(s_0=0.7)$, the pulse decays to zero, which is natural too, as the stable zero solution has its own attraction basin. Note that for the second (smaller) steady-state pulse, which is expected to play the role of the separatrix between the attraction basins of the stable pulse and zero solution, the perturbation theory predicts, in the same case, the amplitude $\tilde{A}_{\mathrm{anal}}\approx 1.26$, so it seems quite natural that the initial pulses with $A_0=1.96$ and $A_0=0.89$ relax, respectively, to the stable pulse and to zero.

For the phenomenological model, no definite stability region for the pulses has been found above; instead, the stability of the zero solution at each point inside the region bounded by the solid lines in Fig. 6 should be checked separately. For instance, the zero solution is found to be stable at the parameter point $(\alpha=0.2,\ \Gamma=0.55)$ in Fig. 6. At this point, the two physical roots of Eq. (35) are 2.073 and 1.786. The amplitudes of the corresponding solitons predicted by the perturbation theory are $\tilde{A}_{\mathrm{anal}}\approx 4.72$ and $\tilde{A}_{\mathrm{anal}}\approx 1.81$, respectively; the one with the larger amplitude may be stable according to the general principle formulated above. Simulations demonstrate that, indeed, there is a stable 2D pulse with a shape very close to the predicted one, all the initial states in the form of the conservative-model lump, whose amplitude exceeds 2.0, relaxing to this stable SP. Figure 8 shows the shapes of the u and v components of the SP at $t=400$, generated by the initial configuration taken as the soliton of the zero-order system (21) with the amplitude 13.33. The amplitude of the thus obtained u components is about 5.14, the discrepancy with the analytical prediction being less than 9%.

It was also found that all the initial lump pulses whose amplitude is less than 1.6 decay to zero. This fact complies well with the expectation that the second (smaller-amplitude) steady-state pulse, predicted by the perturbation theory, whose amplitude is 1.81, ought to be the separatrix between the attraction basins of the stable pulse and zero solution in the phenomenological system.

The simulations also show that, even if the zero background is unstable, lump-type SPs may be quite stable when the integration domain (supplemented by periodic boundary conditions in both x and y) is not very large. An explanation given in Ref. [8] for a similar ''overstability'' effect found in the 1D model, which is based on the suppression of nascent perturbations by the pulse periodically traveling around the domain, applies to the 2D case as well.

Numerical simulations of interactions between two or more stable lump-type SPs were also carried out. Unlike the 1D system (1), (2), in which stable bound states of two and three pulses were easily found [8] (and unlike a physically relevant 2D model of a different type, which follows the pattern of the Zakharov-Kuznetsov equation [20] and will be considered elsewhere), bound states of pulses have *not* been found in both the physical and phenomenological systems based on the KP-I equation. Actually, this fact may be benign for the experimental observation of the 2D pulses, as there will be no competition with multihumped structures that might render the picture much more complex.

VI. CONCLUSION

In this work, we have extended the 1D stabilized Kuramoto-Sivashinsky system to the 2D case. The 2D model is, quite naturally, based on the Kadomtsev-Petviashvili (KP) equation supplemented by loss and gain terms. A model with the dissipative terms of the Newell-Whitehead-Segel type was derived in a consistent way for the weakly 2D case. The derivation was outlined for a particular physical system, a downflowing liquid film carrying a surfactant that diffuses along its surface. It was argued that, in fact, the derived model is generic, therefore it also applies, for instance, to the description of interfacial waves in a two-layered flowing film and double-front flames in a gaseous mixture. Another model, which may apply to a more isotropic situation, was put forward as a semiphenomenological one. Both models consist of the generalized KP-I equation with gain and loss terms linearly coupled to an extra equation of the advection-diffusion type. The additional linear equation stabilizes the system's zero solution, thus paving the way to the existence of completely stable 2D localized solitary pulses, which are objects of an obvious physical interest. Treating the losses and gain as small perturbations, and employing the balance equation for the net field momentum, we have found that the condition of the equilibrium between the losses and gain may select two steady-state 2D solitons from their continuous family, which was found, in an exact analytical form, in the absence of the loss and gain (the exact solitons are similar to the ''lump'' solutions of the KP-I equation). When the zero solution is stable and, simultaneously, two lump-type pulses are picked up by the balance equation for the momentum, the pulse with the larger value of amplitude is expected to be stable in the infinitely long system, while the other pulse must be unstable, playing the role of a separatrix between the attraction domains of the stable pulse and zero solution. These predictions have been completely confirmed by direct simulations for both the physical and phenomenological systems. Another noteworthy finding is that, unlike their 1D counterpart, both KP-based 2D systems do not generate stable bound states of pulses.

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